Journal of Applied Mechanics and Technical Physics, Vol. 47, No. 5, pp. 736–746, 2006

CRITICAL EQUILIBRIUM STATE OF A BUSHING IN A CONTACT PAIR WITH CRACKS POSSESSING PLASTIC TIP ZONES

V. M. Mirsalimov

UDC 539.375

The problem of mechanics of contact fracture is considered for a bushing in a friction pair. It is assumed that multiple reciprocating motion of the plunger leads to fracture of the bushing material owing to friction caused by contact interaction and accompanied by the joint effect of loading and temperature. It is assumed that there are several arbitrarily located straight-line cracks with tip zones near the contact surface of the bushing. The stress state of the bushing is examined in the presence of regions where the crack faces (or some part of them) come into contact.

Key words: *bushing in a contact pair, fracture, crack, plastic flow.*

1. Experience in exploitation of bushing–plunger friction pairs shows that the fracture of the bushing material occurs in contact spots in thin surface layers owing to formation of microcracks. Therefore, at the stage of design of the structure of moving joints, one has to take into account that cracks may appear in individual structural elements (bushing, plunger) and to perform the critical analysis of the elements of the contact pair to ensure that the tentative initial cracks located in the most adverse manner will not reach a critical size and will not cause fracture during the expected service life. The size of the initial minimum crack should be considered as a design characteristic of the material.

Let us consider the stress–strain state of the bushing in operation of the contact pair. Let the bushing in the vicinity of the friction surface contain N straight-line cracks with tip zones of length $2l_k$ ($k = 1, 2, \ldots, N$).

We place the origins of the local coordinate systems $x_kO_ky_k$ into the crack centers; the axes O_kx_k of these coordinate systems coincide with the crack directions and form the angles α_k with the axis Ox (Fig. 1). A high concentration of stresses in the vicinity of the crack tip sometimes leads to softening of the material surrounding the crack. This can be manifested in formation of plastic-flow regions. An analysis of experimental data and conditions of equilibrium and crack development with allowance for interaction between the crack faces and softening zones suggests a model of a crack with a tip zone (pre-fracture zone) with a constant-stress plastic flow. Some publications consider crack models that imply the presence of a constant-stress plastic flow in tip zones commensurable in size with the crack length (see the review in [1]).

Let us identify the crack segments d_{1k} and d_{2k} (tip zones) adjacent to the crack tips, which are regions of a constant-stress plastic flow for the examined material. Interaction of the crack faces in the tip zones is modeled by introducing plastic slip lines (degenerate plasticity bands) between the crack faces. The size of the tip zones depends on the type of the material.

As the tip zones and the thickness of the plastic-flow zone are small as compared with the remaining (elastic) portion of the bushing, they can be replaced by cuts whose surfaces interact in accordance with a certain law and prevent crack opening.

Azerbaijan Technical University, Baku, Azerbaijan AZ1073; irakon63@hotmail.com. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 5, pp. 145–156, September–October, 2006. Original article submitted September 15, 2004; revision submitted May 4, 2005.

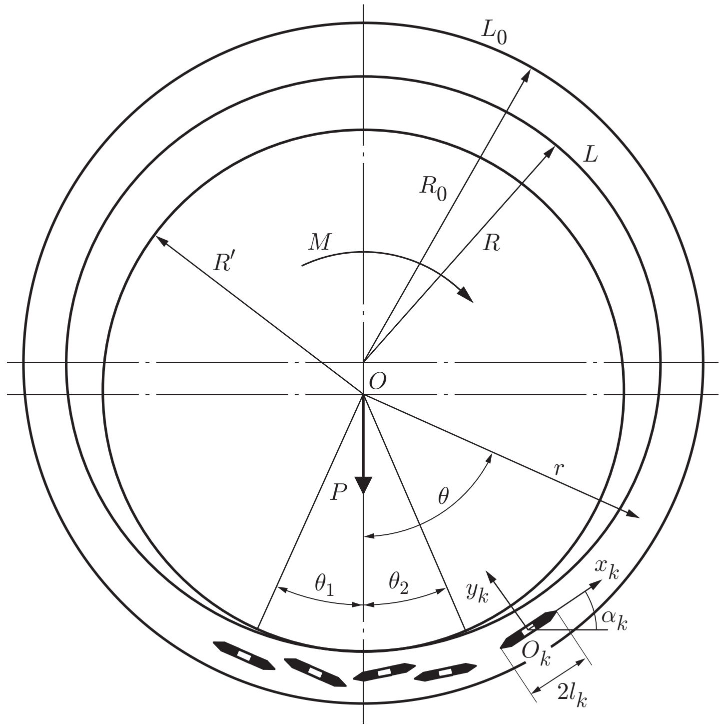

Fig. 1. Numerical scheme of the problem of contact-fracture mechanics.

The contact pressure and friction forces (external loads) acting on the bushing in the tip zones connecting the crack faces generate normal forces $\sigma_{y_k}(x_k) = \sigma_s$ and tangential forces $\tau_{x_k y_k}(x_k) = \tau_s$. Thus, the normal stresses σ_s and tangential stresses τ_s are applied to the crack faces in the tip zones. The sizes of the tip zones are unknown in advance and have to be determined in solving the considered problem of fracture mechanics. The crack faces outside the tip zones (in the internal region of the crack) are free from loading.

We introduce a polar coordinate system at the center of concentric circumferences L and L_0 with radii R and R_0, respectively.

The boundary conditions for the problem considered have the following form:

— for $r = R$,

$$\sigma_r = -p(\theta), \qquad \tau_{r\theta} = -fp(\theta) \tag{1.1}$$

on the contact area and

$$\sigma_r = 0, \qquad \tau_{r\theta} = 0 \tag{1.2}$$

outside the contact area;

— for $r = R_0$,

$$v_r = 0, \qquad v_\theta = 0; \tag{1.3}$$

— on the crack faces,

$$\begin{aligned} &\sigma_{y_k} = 0, \qquad \tau_{x_k y_k} = 0 \qquad \text{on} \quad L'_k \qquad (k = 1, 2, \ldots, N), \\ &\sigma_{y_k} = \sigma_s, \qquad \tau_{x_k y_k} = \tau_s \qquad \text{on} \quad L''_k. \end{aligned} \tag{1.4}$$

Here f is the friction coefficient of the contact pair, v_r and v_θ are the radial and tangential components of the displacement vector, σ_r and $\tau_{r\theta}$ are the stress-tensor components, L'_k are the free faces of the kth crack, and L''_k are the faces of the kth crack with the tip zones where the plastic flow occurs. It is assumed in (1.1)–(1.4) that the tangential (shear) stress in the contact zone is related to the normal pressure $p(\theta)$ by the Coulomb–Amonton law.

The contact pressure is unknown in advance and has to be determined in solving the contact-fracture mechanics problem. To solve this problem, one has to jointly solve the wear-contact problem of plunger pressing into the bushing surface and the problem of fracture mechanics.

Let a plunger with mechanical characteristics G_1 and μ_1 be pressed into the inner surface of a bushing with mechanical characteristics G (shear modulus) and μ (Poisson's ratio) at a certain point unknown in advance. The outer surface of the bushing is assumed to be supported by a rigid shell. The problem is solved under conditions of planar deformation.

The condition relating the bushing and plunger displacements is written as follows [2, 3]:

$$v_1 + v_2 = \delta(\theta) \qquad (\theta_1 \leqslant \theta \leqslant \theta_2). \tag{1.5}$$

Here $\delta(\theta)$ is the settlement of the points of the bushing and plunger surfaces, which is determined by the shape of the inner surfaces of the bushing and plunger and also by the magnitude of the pressing force P; $\theta_2 - \theta_1$ is the magnitude of the contact angle (area).

The shear (friction) forces $\tau_{r\theta}(\theta, t)$ favor heat release in the contact zone. The total amount of heat released per unit time is proportional to the power of friction forces, and the amount of heat released at the point of the contact zone with the coordinate θ is

$$Q(\theta, t) = Vfp(\theta, t)$$

(V is the velocity of plunger motion with respect to the bushing, averaged over the period).

The total amount of heat $Q(\theta, t)$ is spent on increasing the temperature of the bushing $Q_b(\theta, t)$ and plunger $Q_1(\theta, t)$:

$$Q = Q_b + Q_1.$$

For radial motion of the bushing, we have

$$v_1 = v_{1y} + v_{1u}, \tag{1.6}$$

where v_{1y} are the radial thermoelastic displacements of the points of the contact surface of the bushing and v_{1u} are the displacements caused by bushing-surface wear.

To simplify the problem, displacements caused by the collapse of microbuldges of the bushing surface are ignored.

In a form similar to Eq. (1.6), we can write a relation for the radial displacement of the plunger v_2.

The wear of the elements of the contact pair is assumed to be abrasive. The velocity of motion of the surface due to bushing-material wear is determined by the formula [3, 4]

$$\frac{dv_{1u}}{dt} = K_b p(\theta, t), \tag{1.7}$$

where K_b is the wear coefficient of the bushing material.

The bushing is heated as a result of its friction on the plunger surface during reciprocating motion. As the frequency of plunger motion is rather high, the problem is considered in a steady formulation.

To determine thermoelastic displacements v_{1y}, we have to find the temperature distribution in the bushing. For this purpose, we solve the heat-conduction problem

$$\Delta T = 0 \tag{1.8}$$

in the bushing,

$$\begin{gathered} A_{T1}\lambda \frac{\partial T}{\partial r} - A_{T2}\alpha_1(T - T_c) = -Q_* \qquad \text{for} \quad r = R, \\ \lambda \frac{\partial T}{\partial r} + \alpha_2(T - T_c) = 0 \qquad \text{for} \quad r = R_0. \end{gathered} \tag{1.9}$$

Here λ is the thermal conductivity of the bushing, Δ is the Laplace operator, α_1 is the coefficient of heat transfer from the inner surface of the bushing, α_2 is the coefficient of heat transfer between the outer surface of the cylinder and the ambient medium at the temperature T_c, Q_* is the amount of heat released due to friction, which is spent on bushing heating ($Q_* = Q_b$ on the contact area and $Q_* = 0$ outside the contact area), A_{T1} is the area of the heat-consuming surface, and A_{T2} is the area of the cooling surface.

The problem of thermoelasticity for determining the displacements of the contact surface of the plunger is formulated in a similar manner.

The values of θ_1 and θ_2 corresponding to the ends of the segment of the bushing–plunger contact are unknown. To determine them, we use a condition [5] that implies that the pressure $p(\theta)$ continuously tends to zero and vanishes when the point θ reaches the boundary of the contact zone:

$$p(\theta_1) = 0, \qquad p(\theta_2) = 0. \tag{1.10}$$

2. The solution of the boundary-value problem of the heat-conduction theory is sought by the method of separation of variables. The distribution of the excess temperature of the bushing $t_b = T - T_c$ is found in the form

$$t_b = C_1 + C_2 \ln r + \sum_{k=1}^{\infty} (C_1^{(k)} r^k + C_2^{(k)} r^{-k}) \cos k\theta + \sum_{k=1}^{\infty} (A_1^{(k)} r^k + A_2^{(k)} r^{-k}) \sin k\theta,$$

where the constants C_1, C_2, $C_1^{(k)}$, $C_2^{(k)}$, $A_1^{(k)}$, and $A_2^{(k)}$ are determined from the boundary conditions of the heat-conduction problem (1.8), (1.9). Being too cumbersome, these formulas are not given here.

To solve the problem of thermoelasticity, we use the thermoelastic potential of displacements [6]. In the problem considered, the thermoelastic potential of displacements for the bushing F is determined by solving the differential equation

$$\Delta F = \frac{1+\mu}{1-\mu}\, \alpha t_b, \tag{2.1}$$

where α is the coefficient of linear thermal expansion.

The solution of Eq. (2.1) is sought in the form

$$F = \sum_{n=0}^{\infty} (f_n \cos n\theta + f_n^* \sin n\theta).$$

For the functions $f_n(r)$ and $f_n^*(r)$, we obtain ordinary differential equations solved by the method of variation of constants. Determining the thermoelastic potential of displacements for the bushing by known formulas [6], we find the corresponding stresses σ_r^1, σ_θ^1, and $\tau_{r\theta}^1$ and displacements v_r^1 and v_θ^1. The resultant stresses and displacements for the bushing do not satisfy the boundary conditions (1.1)–(1.4).

For the bushing, we need to find the second stress–strain state σ_r^2, σ_θ^2, $\tau_{r\theta}^2$, v_r^2, and v_θ^2 to satisfy the boundary conditions (1.1)–(1.4). To determine the second stress–strain state in the bushing, we have the following boundary conditions:

— for $r = R$,

$$\sigma_r^2 = -p(\theta) - \sigma_r^1, \qquad \tau_{r\theta}^2 = -fp(\theta) - \tau_{r\theta}^1 \tag{2.2}$$

on the contact area and

$$\sigma_r^2 = -\sigma_r^1, \qquad \tau_{r\theta}^2 = -\tau_{r\theta}^1 \tag{2.3}$$

outside the contact area;

— for $r = R_0$,

$$v_r^2 = -v_r^1, \qquad v_\theta^2 = -v_\theta^1; \tag{2.4}$$

— on the crack faces,

$$\begin{aligned} \sigma_{y_k}^2 &= -\sigma_{y_k}^1, \qquad & \tau_{x_k y_k}^2 &= -\tau_{x_k y_k}^1 \qquad & &\text{on} \quad L_k', \\ \sigma_{y_k}^2 &= \sigma_s - \sigma_{y_k}^1, \qquad & \tau_{x_k y_k}^2 &= \tau_s - \tau_{x_k y_k}^1 \qquad & &\text{on} \quad L_k''. \end{aligned} \tag{2.5}$$

Using the Kolosov–Muskhelishvili formulas [5], we can write the boundary conditions (2.2)–(2.5) as the boundary-value problem for complex potentials $\Phi(z)$ and $\Psi(z)$ for the bushing.

The complex potentials are sought in the form

$$\Phi(z) = \Phi_1(z) + \Phi_2(z), \qquad \Psi(z) = \Psi_1(z) + \Psi_2(z); \tag{2.6}$$

$$\Phi_1(z)=\sum_{k=-\infty}^{\infty}a_kz^k,\qquad \Psi_1(z)=\sum_{k=-\infty}^{\infty}b_kz^k,$$

$$\Phi_2(z)=\frac{1}{2\pi}\sum_{k=1}^{N}\int\limits_{-l_k}^{l_k}\frac{g_k(t)\,dt}{t-z_k},\qquad \Psi_2(z)=\frac{1}{2\pi}\sum_{k=1}^{N}\mathrm{e}^{-2i\alpha_k}\int\limits_{-l_k}^{l_k}\Big(\frac{\overline{g_k(t)}}{t-z_k}-\frac{\overline{T_k}\,\mathrm{e}^{i\alpha_k}}{(t-z_k)^2}\,g_k(t)\Big)\,dt, \tag{2.7}$$

where $T_k=t\,\mathrm{e}^{i\alpha_k}+z_k^0$; $z_k=\mathrm{e}^{-i\alpha_k}(z-z_k^0)$; $g_k(x_k)$ are the sought functions characterizing the jump in displacements in passing the corresponding crack.

The boundary-value problem for seeking for the complex potentials on the circular boundaries can be presented in the following form:

$$\begin{aligned}&\Phi_1(\tau)+\overline{\Phi_1(\tau)}-\mathrm{e}^{2i\theta}[\bar\tau\Phi_1'(\tau)+\Psi_1(\tau)]=X(\theta)-(\sigma_r^1-i\tau_{r\theta}^1)-(f_1-if_2),\\&\Phi_1(\tau_0)-k_b\overline{\Phi_1(\tau_0)}-\mathrm{e}^{2i\theta}[\bar\tau_0\Phi_1'(\tau_0)+\Psi_1(\tau_0)]=-2G(v_r^1-iv_\theta^1)'-(f_3-if_4).\end{aligned} \tag{2.8}$$

Here $k_b=3-4\mu$, $\tau=R\exp{(i\theta)}$, $\tau_0=R_0\exp{(i\theta)}$, and $X(\theta)=-(1-if)p(\theta)$ on the contact area and $X(\theta)=0$ outside the contact area;

$$f_1-if_2=\Phi_2(\tau)+\overline{\Phi_2(\tau)}-\mathrm{e}^{2i\theta}[\bar\tau\Phi_2'(\tau)+\Psi_2(\tau)],$$

$$f_3-if_4=\Phi_2(\tau_0)-k_b\overline{\Phi_2(\tau_0)}-\mathrm{e}^{2i\theta}[\bar\tau_0\Phi_2'(\tau_0)+\Psi_2(\tau_0)].$$

To solve the boundary-value problem (2.8) with respect to the potentials $\Phi_1(z)$ and $\Psi_1(z)$, we use the method of power series. For this purpose, we expand the right sides of conditions (2.8) into the Fourier series. After some transformations, we obtain an infinite linear algebraic system with respect to the coefficients a_k and b_k, whose solution is written as

$$a_0=\frac{(A_0+A_0'+D_0)R^2-(F_0+D_0')R_0^2}{2R^2-(1-k_b)R_0^2},\qquad a_{-1}=\frac{(A_1+A_1'+D_1)R}{1+k_b},$$

$$b_{-2}R^{-2}=2a_0-A_0-A_0'-D_0,\qquad b_{-1}=-\frac{k_b(\overline{A_1}+\overline{A_1'}+\overline{D_1}\,)R}{1+k_b},$$

$$a_k=\frac{(1+k)(R_0^2-R^2)B_k-\overline{B}_{-k}(R^{-2k+2}+k_bR_0^{-2k+2})}{(1-k^2)(R_0^2-R^2)^2-(R^{-2k+2}+k_bR_0^{-2k+2})(R^{2k+2}+k_bR_0^{2k+2})}\qquad (k=\pm2,\pm3,\ldots),$$

$$B_k=(F_k+D_k')R_0^{-k+2}-(A_k+A_k'+D_k)R^{-k+2},$$

$$a_1=\frac{2(A_1+A_1'+D_1)R(R_0^2-R^2)}{(1-k_b)(R^4+k_bR_0^4)}-\frac{\overline{B}_{-1}}{R^4+k_bR_0^4},$$

$$b_{k-2}R^{k-2}=(1-k)a_kR^k+\bar a_{-k}R^{-k}-(A_k+A_k'+D_k).$$

The following expansions are used here:

$$X(\theta)=\sum_{k=-\infty}^{\infty}A_k\,\mathrm{e}^{ik\theta},\qquad -(\sigma_r^1-i\tau_{r\theta}^1)=\sum_{k=-\infty}^{\infty}A_k'\,\mathrm{e}^{ik\theta},\qquad -(f_1-if_2)=\sum_{k=-\infty}^{\infty}D_k\,\mathrm{e}^{ik\theta},$$

$$-2G(v_r^1-iv_\theta^1)'=\sum_{k=-\infty}^{\infty}F_k\,\mathrm{e}^{ik\theta},\qquad -(f_3-if_4)=D_k'\,\mathrm{e}^{ik\theta}.$$

The right sides of these formulas contain integrals of the sought functions $g_k(t)$ and the coefficients of expansion of the contact pressure $p(\theta)$.

The functions (2.6) and (2.7) should satisfy the boundary conditions at the crack faces (2.5). From this condition, we obtain a system of N singular integral equations with respect to the unknown functions $g_k(x_k)$:

$$\sum_{k=1}^{N} \int_{-l_k}^{l_k} \left[R_{nk}(t,x)g_k(t) + S_{nk}(t,x)\overline{g_k(t)}\right] dt = \pi[f_n(x) + f]; \tag{2.9}$$

$$|x| \leqslant l_n \qquad (n = 1, 2, \ldots, N),$$

$$f_n(x) = -(\sigma^1_{y_n} - i\tau^1_{x_n y_n}) - [\Phi_1(x_n) + \overline{\Phi_1(x_n)} + x_n \overline{\Phi_1'(x_n)} + \overline{\Psi_1(x_n)}],$$

$$f = \begin{cases} \sigma_s - i\tau_s & \text{on} \quad L'', \\ 0 & \text{on} \quad L', \end{cases} \qquad L' = \sum_{k=1}^{N} L'_k, \qquad L'' = \sum_{k=1}^{N} L''_k.$$

Here x, t, and l_n are dimensionless variables normalized to R; the values of R_{nk} and S_{nk} are determined from relations given in [7].

The system of singular integral equations for internal cracks should be supplemented by the equalities

$$\int_{-l_k}^{l_k} g_k(t)\, dt = 0 \qquad (k = 1, 2, \ldots, N). \tag{2.10}$$

The system of complex singular integral equations (2.9) under the above-indicated conditions (2.10) reduces to a system of $N \times M$ complex algebraic equations [7, 8] for $N \times M$ unknowns $g_n(t_m) = v_n(t_m) - iu_n(t_m)$ ($n = 1, 2, \ldots, N$; $m = 1, 2, \ldots, M$):

$$\frac{1}{M} \sum_{m=1}^{M} \sum_{k=1}^{N} l_k \left[g_k(t_m) R_{nk}(l_k t_m, l_n x_r) + \overline{g_k(t_m)} S_{nk}(l_k t_m, l_n x_r)\right] = f_n(x_r) + f, \tag{2.11}$$

$$\sum_{m=1}^{M} g_n(t_m) = 0 \qquad (n = 1, 2, \ldots, N, \quad r = 1, 2, \ldots, M-1).$$

Here $t_m = \cos\left((2m-1)\pi/(2M)\right)$ $(m = 1, 2, \ldots, M)$; $x_r = \cos\left(\pi r/M\right)$ $(r = 1, 2, \ldots, M-1)$. If we pass to complex-conjugate quantities in (2.11), we obtain additional $N \times M$ algebraic equations that contain the unknown sizes of the tip zones d_{1k} and d_{2k} $(k = 1, 2, \ldots, N)$. For this reason, the algebraic systems (2.11) are nonlinear. To construct the missing $2 \times N$ equations determining the sizes of the tip zones, we use the condition of finite stresses at the crack tips. Finite values of stresses at the crack tips are provided by the joint action of the external load and stresses on the crack faces in the tip zones (postulate on elimination of singularities). The postulate of elimination of singularities is equivalent to the condition of the zero final coefficient of stress intensity, which is determined as the difference between the intensity of stresses caused by external forces and the intensity of stresses induced by pressing forces applied at the tip zones of the crack.

Thus, the equations for determining the sizes of the tip zones d_{1k} and d_{2k} have the form

$$\begin{gathered} \sum_{m=1}^{M} (-1)^m g_k(t_m) \cot \frac{2m-1}{4M}\pi = 0 \qquad (k = 1, 2, \ldots, N), \\ \sum_{m=1}^{M} (-1)^{M+m} g_k(t_m) \tan \frac{2m-1}{4M}\pi = 0. \end{gathered} \tag{2.12}$$

Using the thermoelastic potential of displacements, complex functions (2.6), (2.7), Kolosov–Muskhelishvili formulas, and integration of the kinetic equation (1.7) of wear of the bushing material, we can find the displacement of the contact surface of the bushing v_1.

The problem of thermoelasticity for the plunger is considered in a similar manner. Using the solution of this problem and the kinetic equation of wear of the plunger material, we can find the displacement of its contact surface v_2.

The found values of v_1 and v_2 are substituted into the main contact equation (1.5). To substitute the main contact equation by an algebraic equation, the unknown functions of the contact pressure in a sufficiently small time interval are sought in the form of expansions

$$
\begin{gathered}
p(\theta,t) = p_0(\theta) + tp_1(\theta) + \ldots, \\
p_0(\theta) = \alpha_0 + \sum_{k=1}^{\infty}(\alpha_k \cos k\theta + \beta_k \sin k\theta), \\
p_1(\theta) = \alpha_0^1 + \sum_{k=1}^{\infty}(\alpha_k^1 \cos k\theta + \beta_k^1 \sin k\theta), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{gathered}
\tag{2.13}
$$

A small time interval was chosen because the solution with a small initial interval of time is of greatest interest for a designer. A sufficiently large time interval requires the use of another expansion.

Substituting Eq. (2.13) into the main contact equation, we obtain functional equations for consecutive determination of $p_0(\theta)$, $p_1(\theta)$, etc.

To construct a resolving algebraic system for finding the coefficients of the contact-pressure function in both sides of the functional equation of the contact problem, we equate the coefficients at identical trigonometric functions. As a result, we obtain an infinite algebraic system with respect to α_k^0 $(k = 0, 1, 2, \ldots)$, β_k^0 $(k = 1, 2, \ldots)$, α_k^1, β_k^1, etc.

Because of the presence of unknown quantities θ_1 and θ_2, the system of equations of the contact problem is nonlinear.

The right sides of the infinite algebraic systems contain integrals of the unknown functions $g_k(t)$. In other words, system (2.11), (2.12) and the infinite algebraic system with respect to α_k, β_k are related and have to be solved together.

The combined system of equations is nonlinear because of the presence of the unknowns θ_1, θ_2, d_{1k}, and d_{2k} $(k = 1, 2, \ldots, N)$. To solve this system, we use the method of consecutive approximations [8]. Let us solve the combined algebraic system for certain values of θ_{1*}, θ_{2*}, d_{1k}^*, and d_{2k}^* $(k = 1, 2, \ldots, N)$ with respect to the unknowns $g_k(t_m)$ $(k = 1, 2, \ldots, N;\ m = 1, 2, \ldots, M)$, α_k, and β_k. For this purpose, we have to solve a linear algebraic system. The values of θ_{1*}, θ_{2*}, d_{1k}^*, and d_{2k}^* $(k = 1, 2, \ldots, N)$ and the found values of the remaining unknowns are substituted into the unused Eqs. (1.10), (2.12). Generally speaking, the values of θ_{1*}, θ_{2*}, d_{1k}^*, and d_{2k}^* and the corresponding values of the remaining unknowns will not satisfy Eqs. (1.10) and (2.12). Therefore, the values of the parameters θ_{1*}, θ_{2*}, d_{1k}^*, and d_{2k}^* $(k = 1, 2, \ldots, N)$ are found by iterations until the last equations of the system (1.10) and (2.12) are satisfied with a prescribed accuracy.

The considered problem of mechanics of contact fracture has many free parameters: various thermophysical and mechanical characteristics of materials, geometric sizes of the bushing and plunger, and velocity of plunger motion. For numerical implementation of the method described above, we performed calculations for a U8-6MA2 two-way slush pump. The resultant quantities obtained were the coefficients of the contact-pressure function, approximate values of the functions $g_k^0(t_m) = v_k^0(t_m) - iu_k^0(t_m)$ at nodal points, and the sizes of the tip zones of the cracks and the contact area. Knowing the contact pressure, we can find the temperature distribution, the stress–strain state, and the wear of the elements of the contact pair.

To determine the limiting equilibrium of the crack tip with plastic-flow tip zones, we use the condition of the limiting (critical) opening near the foundation of the plastic zone. The critical state is assumed to occur at the moment when the following condition is satisfied at the edge of the tip zone of the plastic flow:

$$
\sqrt{u^2(x_0) + v^2(x_0)} = \delta_k.
$$

Here $u(x_0) = u^+ - u^-$, $v(x_0) = v^+ - v^-$, and δ_k is the experimentally determined constant of the material, which characterizes the limiting opening of the crack under given conditions.

With allowance for the above-obtained solution, we have

$$
-\frac{1 + k_b}{2G} \int_{-l_k}^{x_0} g_k(x)\,dx = v_k(x_0, 0) - iu_k(x_0, 0) \qquad (k = 1, 2, \ldots, N).
$$

Using the substitution of the variable, replacing the integral by the sum, and separating the real and imaginary parts, we obtain

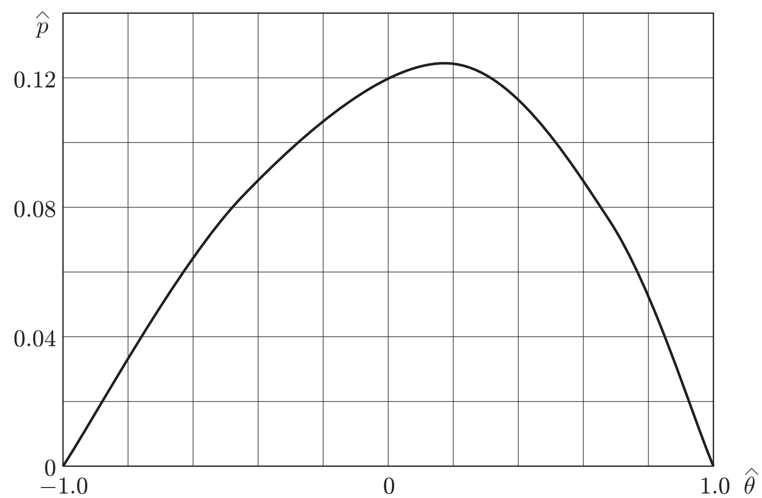

Fig. 2. Contact pressure versus the polar angle for the slush-pump bushing ($V = 0.4$ m/sec).

$$v_k(x_0, 0) = -\frac{1+k_b}{2G}\,\frac{\pi l_k}{M}\sum_{m=1}^{M_{1k}} v_k^0(t_m), \qquad u_k(x_0, 0) = -\frac{1+k_b}{2G}\,\frac{\pi l_k}{M}\sum_{m=1}^{M_{1k}} u_k^0(t_m).$$

Here M_{1k} is the number of nodal points in the interval $(-l_k, x_0)$.

Thus, the condition determining the limiting load at which the crack-tip motion occurs (breakdown of bonds in the plasticity band) is

$$\frac{1+k_b}{2G}\,\frac{\pi l_k}{M}\sqrt{A^2+B^2} = \delta_k, \tag{2.14}$$

where

$$A = \sum_{m=1}^{M_{1k}} v_k^0(t_m), \qquad B = \sum_{m=1}^{M_{1k}} u_k^0(t_m).$$

For the case of one crack, the calculated contact pressure $\hat{p} = p(\theta')R/\Delta E$ is plotted in Fig. 2 as a function of the polar angle $\hat{\theta} = \theta'/\theta_0$ [$\theta' = \theta - \theta_+$, $\theta_0 = (\theta_2 - \theta_1)/2$, and $\theta_+ = (\theta_2 + \theta_1)/2$] for the plunger velocity $V = 0.4$ m/sec. The following values of parameters are used as constants: $2R_0 = 73$ mm, $2R' = 56.7$ mm, $2R = 57$ mm, $f = 0.2$, $E = 1.8 \cdot 10^5$ MPa, $E_1 = 2.1 \cdot 10^5$ MPa, $\mu = 0.25$, $\mu_1 = 0.3$, $K_b = 1.2 \cdot 10^{-10}$, $\Delta = 0.3$ mm, and $\alpha_1 = 45°$.

The contact pressure after the tenth stroke of the plunger was calculated. The greatest values of the contact pressure are normally located in the middle part of the contact surface, depending on the contact angle and friction coefficient. The presence of friction forces in the contact zone shifts the distribution of the contact pressure in the direction opposite to the moment direction.

For a slush-pump bushing, the dimensionless length of the tip zone of plasticity d_{11}/l_1 versus the dimensionless contact pressure p/σ_s is plotted in Fig. 3 for $V = 0.4$ m/sec and different lengths of the crack [$\hat{l} = l_1/(R_0 - R) = 0.2$, 0.3, and 0.4].

The joint solution of the combined nonlinear system of equations and Eq. (2.4) allows us to determine the critical contact pressure as a function of the crack length, sizes of the plastic tip zones and the contact area, and also the values of the sought functions $v^0(t_m)$ and $u^0(t_m)$ ($m = 1, 2, \ldots, M$). Figure 4 shows the dimensionless critical load $p_* = p/\sigma_s$ versus the dimensionless length of the crack $l_* = 8\sigma_s l_1/(\pi E \delta_k)$ in a slush-pump bushing for a velocity $V = 0.4$ m/sec.

3. An analysis of the stress state of the bushing in a constant pair shows that regions of compressing stresses appear in the course of operation of the contact pair, as the plunger is pressed into the bushing surface. We assume that there exist zones where the crack faces (or some part of them) are in contact. We assume that these zones are adjacent to the crack tips, and their sizes are unknown in advance, are commensurable with the crack length, but are smaller than the size of the plastic tip zones.

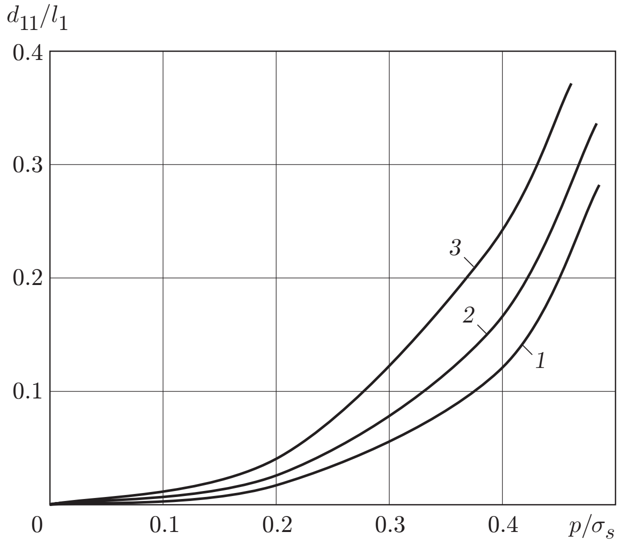

Fig. 3

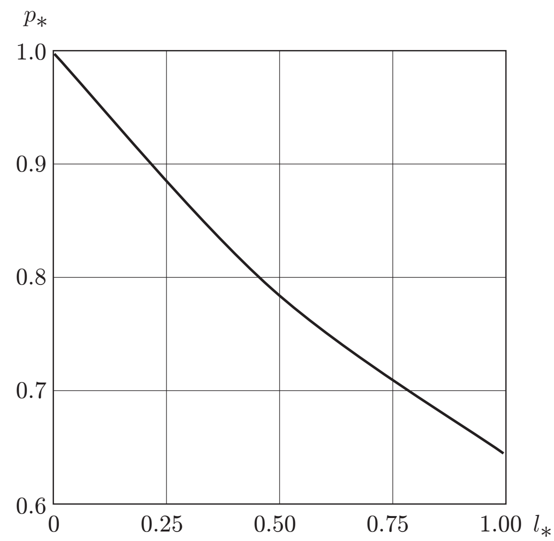

Fig. 4

Fig. 3. Length of the tip zone of the plastic flow versus the contact pressure for the slush-pump bushing ($V = 0.4$ m/sec): $\hat{l} = 0.2$ (1), 0.3 (2), and 0.4 (3).

Fig. 4. Limiting load versus the crack length for the slush-pump bushing.

We consider crack segments of length $\hat{l}_{1k}$ and $\hat{l}_{2k}$ $(k = 1, 2, \dots, N)$ (contact tip zones) adjacent to the crack tips at which the crack faces are in contact. Interaction of the crack faces prevents crack opening.

In the tip zones, where the crack faces are in contact, there arise normal stresses $q_{y_k}(x_k)$ and shear stresses $q_{x_k y_k}(x_k)$. The values of these contact stresses are unknown in advance and have to be determined in solving the boundary-value problem of mechanics of contact fracture. We have to recall that each crack considered in the present case consists of three zones: internal zone and two tip zones. The internal zone of the crack comprises the crack faces free from loads. Two tip zones of the crack are the plastic tip zone $(\hat{l}_{1k}, d_{1k})$, $(\hat{l}_{2k}, d_{2k})$ and the tip zone $(-l_k, \hat{l}_{1k})$, $(\hat{l}_{2k}, l_k)$ where the crack faces are in contact.

The boundary conditions on the crack faces are

$$\begin{aligned}
&\sigma_{y_k} = 0, \qquad &&\tau_{x_k y_k} = 0 \qquad &&\text{on} \quad L' \qquad (k = 1, 2, \dots, N),\\
&\sigma_{y_k} = \sigma_s, \qquad &&\tau_{x_k y_k} = \tau_s \qquad &&\text{on} \quad L'',\\
&\sigma_{y_k} = q_{y_k}, \qquad &&\tau_{x_k y_k} = q_{x_k y_k} \qquad &&\text{on} \quad L'''.
\end{aligned}$$

Here $L''' = \sum\limits_{k=1}^{N} L_k'''$; L_k''' is the kth tip zone in which the crack faces are in contact. The remaining boundary conditions of the problem of contact-fracture mechanics are the same as those in Sec. 2.

To solve the problem posed, we have to repeat the procedure of deriving the basic resolving equations of the problem. The system of $N \times M$ complex algebraic equations for determining $N \times M$ unknowns $g_n^1(t_m) = v_n^1(t_m) - iu_n^1(t_m)$ $(n = 1, 2, \dots, N;\ m = 1, 2, \dots, M)$ acquires the form

$$\begin{gathered}
\frac{1}{M} \sum_{m=1}^{M} \sum_{k=1}^{N} l_k \Big[g_k^1(t_m) R_{nk}(l_k t_m, l_n x_r) + \overline{g_k^1(t_m)} S_{nk}(l_k t_m, l_n x_r) \Big] = f_n(x_r) + f(x_r),\\
\sum_{m=1}^{M} g_n(t_m) = 0 \qquad (n = 1, 2, \dots, N, \quad r = 1, 2, \dots, M-1),
\end{gathered} \tag{3.1}$$

where

TABLE 1

$l_1/(R_0-R)$	$d_{11}/(R_0-R)$	$d_{21}/(R_0-R)$	$\hat{l}_{11}/(R_0-R)$	$\hat{l}_{21}/(R_0-R)$
0.05	0.013	0.017	0.009	0.011
0.10	0.039	0.052	0.031	0.028
0.15	0.067	0.074	0.058	0.069
0.20	0.087	0.0931	0.087	0.076
0.25	0.095	0.108	0.091	0.087

$$f = \begin{cases} 0 & \text{on} \quad L', \\ \sigma_s - i\tau_s & \text{on} \quad L'', \\ q_{y_k} - iq_{x_k y_k} & \text{on} \quad L'''. \end{cases}$$

In the case considered, the right sides of system (3.1) also contain the unknown values of the contact stresses $q_{y_k}(x_k)$ and $q_{x_k y_k}(x_k)$ at the nodal points that belong to the contact tip zones.

The condition determining the unknown contact stresses arising on the crack faces in the contact tip zones is the condition of the absence of crack opening in these zones. In the problem considered, it is more convenient to write this additional condition for the derivative of the opening displacements of the crack faces:

$$g_k^1(x_k) = \frac{2G}{i(1+k_b)} \frac{\partial}{\partial x_k} [u_k^+(x_k,0) - u_k^-(x_k,0) + i(v_k^+(x_k,0) - v_k^-(x_k,0))] = 0. \tag{3.2}$$

Here x_k is the affix of the points of the faces of the contact tip zones of the kth crack.

Requiring conditions (3.2) to be satisfied at the nodal points contained in the tip zones $(-l_k, \hat{l}_{1k})$ and $(\hat{l}_{2k}, l_k)$, we obtain the missing equations for determining the approximate values of the contact stresses $q_{y_k}(t_{m_{1k}})$ and $q_{x_k y_k}(t_{m_{1k}})$ at the nodal points:

$$g_k^1(t_{m_{1k}}) = 0 \qquad (k = 1, 2, \ldots, N; \quad m_{1k} = 1, 2, \ldots, M_{1k}). \tag{3.3}$$

Here M_{1k} is the number of nodal points that belong to the contact tip zones of the kth crack.

To close system (3.1), (3.3), we need additional $2 \times N$ equations determining the tip-zone sizes. From the conditions of finite stresses near the crack tips, we determine the size of the contact tip zones. Writing the conditions of stress finiteness, we obtain the missing $2 \times N$ equations in the form

$$\begin{gathered} \sum_{m=1}^{M} (-1)^m g_k^1(t_m) \cot \frac{2m-1}{4M} \pi = 0 \qquad (k = 1, 2, \ldots, N), \\ \sum_{m=1}^{M} (-1)^{M+m} g_k^1(t_m) \tan \frac{2m-1}{4M} \pi = 0. \end{gathered} \tag{3.4}$$

As the sizes of the contact tip zones are unknown, the system of algebraic equations (3.1), (3.3), and (3.4) is nonlinear. In this case, the combined algebraic system consisting of the resolving system of equations of the contact problem and systems (3.1), (3.3), (3.4) is nonlinear because of the presence of the unknowns θ_1, θ_2, $\hat{l}_{1k}$, and $\hat{l}_{2k}$ $(k = 1, 2, \ldots, N)$. To solve this system, we use the method of consecutive approximations. Solving the combined system allows us to determine the values of the coefficients α_k and β_k of the expansion of the contact-pressure function, the values of the sought functions $g_k^1(t_m)$ at the nodal points, the values of $q_{y_k} - iq_{x_k y_k}$ at the nodal points that belong to the contact tip zones, and the sizes of the contact tip zones.

The parameters $d_{11}/(R_0-R)$, $d_{21}/(R_0-R)$, $\hat{l}_{11}/(R_0-R)$, and $\hat{l}_{21}/(R_0-R)$ versus the crack length $l_1/(R_0-R)$ for the plunger velocity $V = 0.4$ m/sec are summarized in Table 1 for the bushing of the U8-6MA2 slush pump.

The case where one end of certain cracks (or all cracks) reaches the inner surface of the bushing is considered in a similar manner. In this case, the number of conditions in (2.10) is smaller by the number of cracks reaching the bushing surface. For surface cracks, equalities (2.10) are replaced by the conditions of stress finiteness on the face reaching the surface $r = R$.

Modeling of crack growth in the bushing in a contact pair in the course of its operation reduces to a parametric study of a resolving algebraic system of a wear-contact problem, a system of singular integral equations (2.9) and (3.1), equations (2.12), (3.3), and (3.4), and the crack-growth criterion (2.14) for different values of the free parameters of the friction pair.

REFERENCES

1. P. M. Vitvitskii, V. V. Panasyuk, and S. Ya. Yarema, “Plastic strains near the crack tip and fracture criteria: Review,” *Problem. Prochn.*, No. 2, 3–19 (1973).
2. L. A. Galin, *Contact Problems of the Theory of Elasticity and Viscoelasticity* [in Russian], Nauka, Moscow (1980).
3. I. G. Goryacheva, *Mechanics of Frictional Interaction* [in Russian], Nauka, Moscow (2001).
4. I. G. Goryacheva and M. N. Dobychin, *Contact Problems in Tribology* [in Russian], Mashinostroenie, Moscow (1988).
5. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Leyden (1975).
6. G. Parkus, *Instationäre Warmes Pannungen*, Wien, Springer-Verlag (1959).
7. V. V. Panasyuk, M. P. Savruk, and A. P. Datsyshin, *Stress Distributions in the Vicinity of Cracks in Plates and Shells* [in Russian], Naukova Dumka, Kiev (1976).
8. V. M. Mirsalimov, *Non-One-Dimensional Elastoplastic Problems* [in Russian], Nauka, Moscow (1987).